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Spectral triples and differential calculi related to the Kronecker foliation

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Abstract

Following the ideas of Connes and Moscovici, we describe two spectral triples related to the Kronecker foliation, whose generalized Dirac operators are related to first and second order signature operators. We also consider the corresponding differential calculi Ω_D, which are drastically different in the two cases. For the second order signature operator we calculate the Chern character of the spectral triple and the Dixmier trace of certain powers of its Dirac operator. As a side-remark, we give a description of a known calculus on the two-dimensional noncommutative torus in terms of generators and relations.

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1. Introduction

In Connes' approach to noncommutative differential geometry, the notion of a spectral triple plays an essential role, see [1]. It encodes the differential and Riemannian structure of the noncommutative space as well as its dimension. From the physical point of view, spectral triples have been used to construct unified field theoretical models, in particular the standard model (see [1,2]), and also models including gravitation [3–5]. From the mathematical point of view, only a few types of noncommutative spaces have been used in these

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examples: commutative algebras of smooth functions on a manifold [1], finite dimensional algebras (for a classification of spectral triples in this case, see [6,7]) and products of both. In [8], it was shown that it is not straightforward to define spectral triples related to covariant differential calculi on quantum groups. Explicit examples of spectral triples have also been described for the irrational rotation algebra and higher dimensional noncommutative tori [1,9]. For these examples, the data of the triple were chosen according to physical needs or taking advantage of special structures available in the underlying algebra. An important part of the information needed for physical purposes is the explicit form of the differential calculus of a spectral triple. Such calculi have been analysed in the above-mentioned cases [1,2,10,11]. In [12], it has been shown that the extra structure of a finitely generated projective module allows to introduce the graded algebra of differential-form-valued endomorphisms which gives a natural mathematical language to build unified field theoretical models in the spirit of the Mainz-Marseille approach [13]. In [14,15], the notion of spectral triple itself has been modified and enriched using ideas from supersymmetric quantum theory. One arrives at noncommutative structures generalizing classical geometrical structures (Riemannian, symplectic, Hermitian, Kähler, etc. structures). Physical hopes are mainly directed to superconformal field theories (with noncommutative target spaces).

Recently, see [16], Connes and Moscovici have described a method which makes it possible to construct spectral triples in a systematic way for crossed product algebras related to foliations. Let $(M, \mathcal{F})$ be a regular foliation of a smooth manifold M with Euclidean structures on both the corresponding distribution and the normal bundle. There is an associated spectral triple for the crossed product algebra $C^\infty(M) \rtimes \Gamma$, where Γ is a group of diffeomorphisms preserving these structures. The corresponding Dirac operator is a hypoelliptic operator which is closely related to the signature operator of the foliated manifold. This signature operator is a modification of the standard signature operator in differential geometry, see [17]. The explicit form of the spectral triple makes it then possible to calculate its Chern character and also the Dixmier trace of hypoelliptic operators.

In this paper, we construct explicitly two spectral triples related to the Kronecker foliation. We choose as diffeomorphism group, the group $\mathbb{R}$ which defines the foliation by its action on $\mathbb{T}^2$ and obviously preserves natural translation invariant Euclidean structures. Thus, we arrive at the algebra $C^\infty(\mathbb{T}^2) \rtimes \mathbb{R}$, whose C version is known to be Morita equivalent to the irrational rotation algebra (noncommutative torus), see [9,18]. The Dirac operator of the first spectral triple (which has dimension 2) is closely related to the ordinary signature operator on $\mathbb{T}^2$. For the construction of the second triple (of dimension 3) we follow the strategy proposed in [16]. The corresponding signature operators and henceforth also the Dirac operators can be diagonalized explicitly in both cases. Then we pass to the differential calculi associated to the spectral triples constructed before. It turns out that for the triple related to the first order signature operator the differential calculus can be completely determined. Restricted to $C^\infty(\mathbb{T}^2)$ it projects down to the de Rham calculus on $\mathbb{T}^2$. The analysis of the differential calculus for the second triple turns out to be much more involved. We show that for the restriction of this triple to the subalgebra $C^\infty(\mathbb{T}^2)$ (i.e. choosing the trivial diffeomorphism group) the corresponding one forms give just the universal calculus on $C^\infty(\mathbb{T}^2)$. Then we pass to the explicit calculations of the Chern character of the spectral triple and the Dixmier trace of its Dirac operator.

In Appendix B we have added the explicit description of the differential calculus for the spectral triple related to the irrational rotation algebra, see [1,9], which has properties similar to the calculus associated to the linear signature operator.

2. The spectral triple related to a foliation

For the convenience of the reader, we recall here the definition of a spectral triple and the differential calculus related to such a triple [1,9].

Definition 1. A spectral triple $(A, \mathcal{H}, D)$ consists of a $*$-algebra A, a Hilbert space $\mathcal{H}$ and an unbounded operator D on $\mathcal{H}$, such that:

(i) A acts by a $*$-representation π in the algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$,
(ii) the commutators $[D, \pi(a)]$, $a \in A$, are bounded and
(iii) the operator D has discrete spectrum with finite multiplicity.

A spectral triple is said to have dimension n, if the eigenvalues (with multiplicity) μ_k of $|D|$ fulfil $\lim_{k\to\infty}(\mu_k/k^{1/n}) = C \neq 0$.

We will have no need to refer to gradings or real structures usually included in the definition of a spectral triple, and also not to more general notions of dimension.

The representation π of A in $B(\mathcal{H})$ can be extended to a representation $\pi^* : \Omega(A) \to B(\mathcal{H})$ of the universal differential calculus $\Omega(A)$ by

$$\pi^n\left(\sum_k a_0^k\, \mathrm{d}a_1^k \cdots \mathrm{d}a_n^k\right) = \sum_k \pi(a_0^k)[D, \pi(a_1^k)] \cdots [D, \pi(a_n^k)].$$

If $J_0 := \oplus_n \ker \pi^n$, then $J := J_0 + \mathrm{d}J_0$ is a differential ideal, and one arrives at the differential calculus $\Omega_D(A)$,

$$\Omega_D^n(A) := \frac{\Omega^n(A)}{J^n}.$$

Note that, if π is faithful, there are isomorphisms

$$\Omega_D^1(A) \simeq \pi^1(\Omega^1(A)), \tag{2.1}$$

and

$$\Omega_D^2(A) \simeq \frac{\pi^2(\Omega^2(A))}{\pi^2(\mathrm{d}J_0^1)}. \tag{2.2}$$

Now we review shortly the procedure given in [16], which relates a spectral triple to a regular foliation of a smooth manifold. Let M be a compact manifold with a foliation given by an integrable distribution $V \subset TM$. The normal bundle of the foliation is $N := TM/V$, with canonical projection $\rho : TM \to N$. Assume further that both V and N are equipped with Euclidean fibre metrics and with an orientation (i.e. there are distinguished nowhere vanishing sections ω_V, ω_N of the exterior bundles $\bigwedge^v V$, $\bigwedge^n N$ ($v = \dim V$, $n =$

dim N)). Furthermore, ω_V and ω_N also define a nonvanishing section of $\bigwedge^v V^* \otimes \bigwedge^n N^* \simeq \bigwedge^{v+n} T^*M$, i.e. a volume form on M. The bundle of interest for us is

$$E = \bigwedge V^*_{\mathbb{C}} \otimes \bigwedge N^*_{\mathbb{C}}.$$

Obviously, the metrics on V and N give rise to Hermitian metrics on $\bigwedge V^*_{\mathbb{C}}$ and $\bigwedge N^*_{\mathbb{C}}$ and thus also on E. The orientations ω_V and ω_N can be mapped by means of the metrics to sections γ_V of $\bigwedge^v V^*_{\mathbb{C}}$ and γ_N of $\bigwedge^n N^*_{\mathbb{C}}$ which can be used, together with the metrics, to define an analogue of the Hodge star on the exterior bundles $\bigwedge V^*_{\mathbb{C}}$ and $\bigwedge N^*_{\mathbb{C}}$. We choose a variant of the $*$-operation such that $*^2_{V_{\mathbb{C}}} = 1$ and $*^2_{N_{\mathbb{C}}} = 1$, i.e. $*_{V_{\mathbb{C}}}$ and $*_{N_{\mathbb{C}}}$ can be considered as $\mathbb{Z}_2$-grading operators (cf. [19]). Thus, the space of sections of E has a natural inner product, and we denote by $\mathcal{H} = L^2(M, E)$ the Hilbert space of square integrable sections of this bundle. From now on, we always consider complexified vector bundles, but omit the subscript $\mathbb{C}$.

In order to construct a generalized Dirac operator, a longitudinal differential $\mathrm{d_L}$ and a transversal differential operator $\mathrm{d_H}$ have to be defined. The differential $\mathrm{d_L}$ is defined canonically by means of the Bott connection [20] given as the partial covariant derivative $\nabla : \Gamma(V) \times \Gamma(N) \to \Gamma(N)$ defined by

$$\nabla_X Y = \rho([X, \tilde{Y}])$$

for $X \in \Gamma(V)$, $Y \in \Gamma(N)$ and $\tilde{Y} \in \Gamma(TM)$ such that $\rho(\tilde{Y}) = Y$. By a standard procedure (using the Leibniz rule and duality) ∇ is extended to a differential $\mathrm{d_L} : \Gamma(E) \to \Gamma(E)$ defined by linear mappings $\Gamma(\bigwedge^k V^* \otimes \bigwedge^l N^*) \to \Gamma(\bigwedge^{k+1} V^* \otimes \bigwedge^l N^*)$,

$$\begin{aligned}\mathrm{d_L}\alpha(X_0, \dots, X_k) = &\sum_{i=0,\dots,k} (-1)^i \nabla_{X_i}(\alpha(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k),\end{aligned}$$

$X_i \in \Gamma(V)$. Since the Bott connection is flat, we have $\mathrm{d_L^2} = 0$.

In order to define a transversal differential operator one has to choose a subbundle $H \subset TM$ complementary to V. This defines a bundle isomorphism $j_H : \bigwedge V^* \otimes \bigwedge N^* \to \bigwedge T^*M$ in the following way. Let us denote by pr^*_V and pr^*_H the projections corresponding to the decomposition $TM^* = V^* \oplus H^*$, by $\rho_H : H \to N$ the restriction of ρ to H and by ρ^*_H its transposed map. Then j_H is defined as the following composition:

$$\bigwedge V^* \otimes \bigwedge N^* \overset{\mathrm{id} \otimes \bigwedge \rho^*_H}{\to} \bigwedge V^* \otimes \bigwedge H^* \overset{\bigwedge \mathrm{pr}^*_V \otimes \bigwedge \mathrm{pr}^*_H}{\to} \bigwedge T^*M \otimes \bigwedge T^*M \overset{\otimes \to \bigwedge}{\to} \bigwedge T^*M,$$

where $\otimes \to \bigwedge$ denotes the replacement of the tensor product by the wedge product. Now, the transversal operator $\mathrm{d_H}$ is obtained from the exterior differential d by transporting with j_H and projecting to a certain homogeneous component: $\bigwedge V^* \otimes \bigwedge N^*$ has an obvious bigrading, and denoting by $\pi^{(r,s)}$ the projector to the homogeneous component of bidegree (r, s), one defines

$$\mathrm{d_H}\alpha = \pi^{(r,s+1)}(j_H^{-1} \circ \mathrm{d} \circ j_H(\alpha))$$

for $\alpha \in \Gamma(\bigwedge^r V^* \otimes \bigwedge^s N^*)$. The operator $\mathrm{d_H}$ is a graded derivation of the $\mathbb{Z}_2$-graded algebra $\Gamma(\bigwedge V^* \otimes \bigwedge N^*)$.

In a foliation chart, $\mathrm{d_L}$ and $\mathrm{d_H}$ look as follows. Let (x^i, y^k), $i = 1, \ldots, v$, $k = 1, \ldots, n$ be local coordinates of M such that x^i are coordinates on the leaf (foliation chart). The corresponding coordinate vector fields $(\partial/\partial x^i, \partial/\partial y^k)$ form a local frame of TM and $(\partial/\partial x^i)$ a frame of V. The corresponding dual frame of T^*M consists of the differentials $(\mathrm{d}x^i, \mathrm{d}y^k)$. We define $\theta^i \in \Gamma(V^*)$ by $\theta^i(\partial/\partial x^j) = \delta^i_j$ $(i, j = 1, \ldots, v)$. It is immediate from the definition of N that the elements $n_k := (\partial/\partial y^k) + V$ $(k = 1, \ldots, n)$ form a local frame of N. The elements of the corresponding dual frame of N^* are denoted by n^k. Finally, we choose a local frame h_k of the transversal space H. This frame is fixed by assuming $\rho_H(h_k) = n_k$. This leads to

$$h_k = h^i_k \frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^k},$$

with coefficient functions h^i_k characterizing H. Then, the elements $\theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s}$ form a local frame of E, and one can show that $\mathrm{d_L}$ and $\mathrm{d_H}$ are given by the following local formulae:

$$\begin{aligned} &\mathrm{d_L}(\alpha_{i_1\cdots i_r j_1 \cdots j_s} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s}) \\ &\quad = \frac{\partial \alpha_{i_1\cdots i_r j_1 \cdots j_s}}{\partial x^i} \theta^i \wedge \theta^{i_1} \wedge \cdots \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s}, \end{aligned}$$

$$\begin{aligned} &\mathrm{d_H}(\alpha_{i_1\cdots i_r j_1 \cdots j_s} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s}) \\ &\quad = (-1)^r \left(\frac{\partial}{\partial y^k} + h^i_k \frac{\partial}{\partial x^i} \right) \alpha_{i_1\cdots i_r j_1 \cdots j_s} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \otimes n^k \wedge n^{j_1} \wedge \cdots \wedge n^{j_s} \\ &\qquad + \alpha_{i_1\cdots i_r j_1 \cdots j_s} \sum_{t=1}^{r} \frac{\partial h^{i_t}_k}{\partial x^l} \theta^{i_1} \wedge \cdots \wedge \theta^l \wedge \cdots \wedge \theta^{i_r} \otimes n^k \wedge n^{j_1} \wedge \cdots \wedge n^{j_s} \end{aligned} \tag{2.3}$$

(where θ^l at position t replaces θ^{i_t}). The longitudinal differential $\mathrm{d_L}$ acts as a differential in leaf direction, whereas $\mathrm{d_H}$ is a sum of a principal part, which differentiates in transversal direction, and a zero order part. As examples, let us give formulae for $\mathrm{d_H}$ acting on functions, (1, 0)-, (0, 1)- and (1, 1)-forms

$$\begin{aligned} &\mathrm{d_H} f = h_k(f) n^k, \qquad \mathrm{d_H}(\alpha_i \theta^i) = -h_k(\alpha_i)\theta^i \otimes n^k - \alpha_i \frac{\partial h^i_k}{\partial x^j} \theta^j \otimes n^k, \\ &\mathrm{d_H}(\alpha_k n^k) = h_l(\alpha_k) n^l \wedge n^k, \\ &\mathrm{d_H}\left(\alpha_{ik}\theta^i \wedge n^k\right) = -h_l(\alpha_{ik})\theta^i \otimes n^l \wedge n^k - \alpha_{ik} \frac{\partial h^i_l}{\partial x^j} \theta^j \otimes n^l \wedge n^k \end{aligned}$$

$(\mathrm{d_H}(n^k) = 0)$. For the adjoint operators $\mathrm{d_L^*}$ and $\mathrm{d_H^*}$ (in $\mathcal{H}$) it is difficult to write down explicit formulae. One can show

$$\mathrm{d_L^*} \alpha = - *_V \ \mathrm{d_L} *_V + \text{terms of order zero},$$

where $*_V$ is the (partial) Hodge operator related to the Euclidean metric and the orientation of V. Since d_H^* lowers the N^*-degree one has for $\alpha \in \Gamma(\bigwedge^r V^*) \equiv \Gamma(\bigwedge^r V^* \otimes \bigwedge^0 N^*)$

$$\mathrm{d}_\mathrm{H}^* \alpha = 0.$$

Explicit formulae for d_H^* become rather complicated as, e.g., the case of (0, 1)-forms shows

$$\mathrm{d}_\mathrm{H}^*(\alpha_i n^i) = -g_N^{kl}\left(h_k(\alpha_l) - \alpha_m {\Gamma_N}^m{}_{kl} + \alpha_l\left(\frac{\partial h_k^i}{\partial x^i} + \frac{1}{2} g_V^{ij} h_k(g_{V_{ij}})\right)\right), \tag{2.4}$$

where $g_N^{kl} = g_N(n^k, n^l)$, $g_{V_{ij}} = g_V(\partial/\partial x^i, \partial/\partial x^j)$, $g_V^{ij} = g_V(\theta^i, \theta^j)$ are the local components of the fibre metrics (and their duals), and ${\Gamma_N}^m{}_{kl}$ are the Christoffel symbols corresponding to $g_{N_{kl}}$.

In [16], using d_H and d_L, two differential operators were introduced by

$$Q_\mathrm{L} = \mathrm{d}_\mathrm{L}\, \mathrm{d}_\mathrm{L}^* - \mathrm{d}_\mathrm{L}^*\, \mathrm{d}_\mathrm{L}, \qquad Q_\mathrm{H} = \mathrm{d}_\mathrm{H} + \mathrm{d}_\mathrm{H}^*,$$

and the mixed signature operator Q for M, acting on a form with N-degree ∂_N, was defined by

$$Q = Q_\mathrm{L}(-1)^{\partial_N} + Q_\mathrm{H}. \tag{2.5}$$

As noted in [16], Q is selfadjoint. Finally, a generalized Dirac operator D is defined as the unique selfadjoint operator such that

$$D|D| = Q. \tag{2.6}$$

If zero is not an element of the spectrum of Q, it is given as

$$D = Q|Q|^{-1/2} = Q(Q^2)^{-1/4}, \tag{2.7}$$

as shows a straightforward argument using the spectral decomposition of Q.

One motivation for choosing a second order longitudinal part is the following: the index of the signature operator should not depend on the choice of the transversal subbundle H. Usually, the index of a pseudodifferential operator only depends on its principal symbol. However, as follows from the local formulae (2.3) and (2.4), the principal part of Q_H explicitly depends on H, the dependence being in the coefficients of the partial derivatives with respect to leaf coordinates. It turns out that one can get rid of this dependence by introducing a modified notion of pseudodifferential operators ($\psi DO'$) which assigns a degree 2 to transversal coordinates and a degree 1 to longitudinal ones. To have a contribution also from Q_L, one has to pass to a second order operator. In [16], a homotopy argument was given to show that this does not affect the longitudinal signature class.

Let Γ be any group of diffeomorphisms of M which preserves the distribution V and the Euclidean metrics on both V and N. Then $\psi \in \Gamma$ acts via the pull back as unitary operator U_ψ^* on $\mathcal{H}$, whereas functions from $C^\infty(M)$ act there as multiplication operators. The crossed product algebra $\mathcal{A} := C^\infty(M) \rtimes \Gamma$ can be defined as the $*$-subalgebra of $B(\mathcal{H})$ generated by these two types of operators. Due to $U_\psi^* f = (f \circ \psi) U_\psi^*$ every element of $\mathcal{A}$ is a finite sum of elements fU_ψ^*. Then we have the following theorem, see [16].

Theorem 1. *$(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple of dimension $v + 2n$.*

Starting from the spectral triple one may now calculate several quantities related to it, e.g., its (odd) Chern character $\mathrm{ch}_*(\mathcal{A}, \mathcal{H}, D)$ and the Dixmier trace of powers of the Dirac operator $\mathrm{Tr}_\omega(D)$ (cf. [1,21]). With

$$F = \frac{D}{|D|}, \tag{2.8}$$

we have for the Chern character

$$\mathrm{ch}_*(\mathcal{A}, \mathcal{H}, D)(a^0, \dots, a^m) = \lambda_m \,\mathrm{Tr}'(a^0[F, a^1][F, a^2]\cdots[F, a^m]) \tag{2.9}$$

for all $a^0, \dots, a^m \in \mathcal{A}$ ($m \geq v + 2n$, m odd), with

$$\mathrm{Tr}'(T) = \tfrac{1}{2}\mathrm{Tr}(F(FT + TF)), \tag{2.10}$$

and

$$\lambda_m = \sqrt{2\mathrm{i}}(-1)^{m(m-1)/2}\Gamma(\tfrac{1}{2}m + 1). \tag{2.11}$$

The extension of the Dixmier trace Tr_ω to any $\psi DO'$ is given by

$$\mathrm{Tr}_\omega(T) = \frac{1}{v + 2n}\int_M c(x) \tag{2.12}$$

with

$$c(x) = \frac{1}{(2\pi)^{-(n+v)}(v + 2n)}\int_{\|\xi\|'=1} \mathrm{tr}\,\sigma_{-(v+2n)}(x, \xi) i_e \,\mathrm{d}\xi, \tag{2.13}$$

where tr is the usual matrix trace (T is in general an operator on sections of a vector bundle) and $\sigma_{-(v+2n)}(x, \xi)$ is the homogeneous part of order $-(v+2n)$ in the asymptotic expansion of T in the $\psi DO'$ calculus (see [16]). Here e is the generator of the flow

$$F_s(\xi_v, \xi_n) = (\mathrm{e}^s \xi_v, \mathrm{e}^{2s}\xi_n),$$

$i_e\,\mathrm{d}\xi$ denotes the contraction of $\mathrm{d}\xi$ by e and the integral is taken over the (modified) cosphere bundle over M.

3. Spectral triples for the Kronecker foliation

3.1. The crossed product algebra for the Kronecker foliation

Let us start with some conventions and notations. We consider the two-torus as the quotient $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. Thus, we have natural local coordinates $0 < \vartheta_1, \vartheta_2 < 2\pi$. Consider the $\mathbb{R}$-manifold $(\mathbb{T}^2, \mathbb{R}, \psi)$, with group action

$$\psi : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2,$$

given by

$$\psi((\vartheta_1, \vartheta_2), t) = (\vartheta_1 + at, \vartheta_2 + bt)$$

with $a, b \in \mathbb{R}$ such that $a > 0$, $a^2 + b^2 = 1$ and $\theta = b/a$ being irrational. The foliation of $\mathbb{T}^2$

by the orbits of ψ is called the Kronecker foliation. It is well known, see [22], that each leaf of this foliation is diffeomorphic to $\mathbb{R}$ and lies dense in $\mathbb{T}^2$. The coordinate transformation

$$x = a\vartheta_1 + b\vartheta_2, \qquad y = b\vartheta_1 - a\vartheta_2$$

is orthogonal and leads to coordinates (x, y) of a foliation chart. In these coordinates, $\mathbb{R}$ acts as follows:

$$\psi((x, y), t) = (x + t, y).$$

To be more precise, this is the lifted action of $\mathbb{R}$ on $\mathbb{R}^2$, applied to global coordinates (x, y) obtained from global coordinates $(\vartheta_1, \vartheta_2)$ by the orthogonal transformation. The fact that each leaf of the foliation lies dense in $\mathbb{T}^2$ implies immediately the following lemma.

Lemma 1. *All metrics g_V and g_N, invariant under the above $\mathbb{R}$-action, are constant in any coordinates affine with respect to ϑ_1 and ϑ_2.*

We fix the constants by

$$g_V\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g_N\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 1. \tag{3.1}$$

It is well known, see [23], that associated to the action of a locally compact group K on a manifold M there is a transformation groupoid G. For the Kronecker foliation, we have $G = \mathbb{T}^2 \times \mathbb{R}$ with range and source maps r and s given by

$$r(p, t) = \psi(p, t), \qquad s(p, t) = p,$$

$p \in \mathbb{T}^2$, the space of units being $\mathbb{T}^2$. The associated crossed product algebra

$$\mathcal{O} := \mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$$

is the $*$-algebra generated by the unitary operators U_1, U_2 and V_t acting in the Hilbert space $L^2(\mathbb{T}^2)$ given by

$$\begin{aligned} &(U_1\xi)(\vartheta_1, \vartheta_2) = \mathrm{e}^{\mathrm{i}\vartheta_1} \cdot \xi(\vartheta_1, \vartheta_2), \qquad (U_2\xi)(\vartheta_1, \vartheta_2) = \mathrm{e}^{\mathrm{i}\vartheta_2} \cdot \xi(\vartheta_1, \vartheta_2), \\ &(V_t\xi)(\vartheta_1, \vartheta_2) = \xi(\vartheta_1 + at, \vartheta_2 + bt) \end{aligned} \tag{3.2}$$

$\forall \xi \in L^2(T^2)$. Let $e_{kl} = \mathrm{e}^{\mathrm{i}(k\vartheta_1 + l\vartheta_2)}$ $(k, l \in \mathbb{Z})$ be the basis of trigonometric polynomials of $L^2(\mathbb{T}^2)$. Obviously, from (3.2) it follows that:

$$U_1 e_{kl} = e_{k+1,l}, \qquad U_2 e_{kl} = e_{k,l+1}, \qquad V_t e_{kl} = \mathrm{e}^{\mathrm{i}(ak+bl)t}\, e_{kl}. \tag{3.3}$$

It is now immediate to show the following proposition.

Proposition 1. *The unitary operators U_1, U_2, and V_t satisfy*

$$U_1 U_2 = U_2 U_1, \tag{3.4}$$

$$V_t U_1 = \mathrm{e}^{\mathrm{i}at}\, U_1 V_t, \tag{3.5}$$

$$V_t U_2 = \mathrm{e}^{\mathrm{i}bt}\, U_2 V_t, \tag{3.6}$$

$$V_t V_s = V_{t+s}, \quad t, s \in \mathbb{R}. \tag{3.7}$$

Remark 1. For rational $a/b = m/n$, m, n relative prime, there is an additional relation

$$V_{2\pi\sqrt{m^2+n^2}} = V_0 = 1,$$

$2\pi\sqrt{m^2+n^2}$ is the smallest value of t such that $V_t = V_0 = 1$ and any other such t is an integer multiple of it.

Proposition 2. *The $*$-algebra $\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$ is isomorphic to $\mathbb{C}\langle u_1, u_2, v_t\rangle/J$, where $\mathbb{C}\langle u_1, u_2, v_t\rangle$ is the free associative unital $*$-algebra generated by u_1, u_2 and v_t, $t \in \mathbb{R}$, and J is the $*$-ideal generated by (3.4)–(3.7) and unitarity conditions for the generators.*

Proof. By universality of $\mathbb{C}\langle u_1, u_2, v_t\rangle/J$, there exists a homomorphism π of this algebra onto $\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$ sending the corresponding generators onto each other. Now, by (3.4)–(3.7), a general element a of $\mathbb{C}\langle u_1, u_2, v_t\rangle/J$ is a linear combination of the monomials $v_{t_j}u_1^k u_2^l$, for some $t_j \in \mathbb{R}$ (with $j \neq j' \Rightarrow t_j \neq t_{j'}$), $k, l \in \mathbb{Z}$, $a = \sum a_{jkl} v_{t_j} u_1^k u_2^l$ (finite sum). We first show that all V_t are independent. $V_t = \sum_j b_j V_{t_j}$ is equivalent to $1 = \sum b_j\, \mathrm{e}^{\mathrm{i}(ka+bl)(t_j-t)}\ \forall k, l$. Since a/b is irrational, $\{ak+bl | k, l \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, and one can conclude $1 = \sum b_j\, \mathrm{e}^{\mathrm{i}x(t_j-t)}\ \forall x \in \mathbb{R}$. But 1 and $\mathrm{e}^{\mathrm{i}x(t_j-t)}$ are orthogonal as almost-periodic functions, see [24], and therefore linearly independent. Then also $V_t U_1^k U_2^l$ and $V_{t'} U_1^k U_2^l$ ($t \neq t'$) are linearly independent. Since $V_t U_1^k U_2^l$ and $V_{t'} U_1^{k'} U_2^{l'}$ ($(k, l) \neq (k', l')$) shift a different number of steps in the above basis e_{kl} they are obviously independent. In other words, the monomials $V_t U_1^k U_2^l$ constitute a basis in $\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$. If $\pi(a) = \sum a_{jkl} V_{t_j} U_1^k U_2^l = 0$ we have $a_{jkl} = 0$ and π is a bijection. □

In analogy with the definition given before Theorem 1, putting $M = \mathbb{T}^2$ and $\Gamma = \mathbb{R}$, we define the crossed product

$$\mathcal{A} := C^\infty(\mathbb{T}^2) \rtimes \mathbb{R} \tag{3.8}$$

as a $*$-subalgebra of $B(L^2(\mathbb{T}^2))$.

Remark 2. We can introduce a set of seminorms on $\mathcal{O} = \mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$ as follows. Let $\mathcal{O}(\mathbb{T}^2) \subset \mathcal{O}$ be the $*$-subalgebra generated by U_1 and U_2. We define a family $(p_n)_{n\in\mathbb{N}}$ of seminorms on this subalgebra by

$$p_n\left(\sum_{jk} a_{jk} U_1^j U_2^k\right) = \sup_{j,k\in\mathbb{Z}} (1 + |j| + |k|)^n |a_{jk}|.$$

It is well known [25, 22.19.2–22.19.4] that the completion of $\mathcal{O}(\mathbb{T}^2)$ in the corresponding Fréchet topology is $C^\infty(\mathbb{T}^2)$. Now we define seminorms on $\mathcal{O}$ by

$$p_n^{\rtimes}\left(\sum_{jkl} a_{jkl} V_{t_j} U_1^k U_2^l\right) = p_n\left(\sum_{jkl} a_{jkl} U_1^k U_2^l\right).$$

Then it is easy to show that $\mathcal{A}$ is the completion $\mathcal{O}$ with respect to the Fréchet topology defined by the family of seminorms $p_n^{\rtimes}$. To this end, one first notes that every element of

$\mathcal{A}$ is a finite sum of products fV_t, $f \in C^\infty(\mathbb{T}^2)$. By the above, f is a limit of elements f_k of $\mathcal{O}(\mathbb{T}^2)$ with respect to p_n, and by the definition of $p_n^{\rtimes}$ it is obvious that fV_t is the limit of $f_k V_t$.

Let us now describe the Hilbert space of the spectral triple of Theorem 1 for the Kronecker foliation.

Here, both V and N are one-dimensional, with local frames consisting each of one vector $\partial/\partial x$ and $\underline{n} = (\partial/\partial y) + V$, respectively. Let τ and ν denote the corresponding elements of the dual frames. Then $E = \bigwedge V^* \otimes \bigwedge N^*$ consists of four one-dimensional subspaces of elements of degrees $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, with local frames $\mathbf{1}$, τ, ν, $\tau \otimes \nu$, respectively. The natural choice of translation invariant (under the natural action of $\mathbb{R}^2$ on $\mathbb{T}^2$) Euclidean fibre metrics makes these frame elements mutually orthogonal unit vectors in $L^2(\mathbb{T}^2, E)$. We may identify

$$L^2(\mathbb{T}^2, E) = L^2(T^2) \oplus L^2(T^2) \oplus L^2(T^2) \oplus L^2(T^2)$$

with $e_{kl}1 \to (e_{kl}, 0, 0, 0), \ldots, e_{kl}\tau \otimes \nu \to (0, 0, 0, e_{kl})$.

Remark 3. Since the generators act, according to (3.2), componentwise in $L^2(\mathbb{T}^2, E)$, the crossed product algebra of Theorem 1 coincides with (3.8).

We choose the transversal subspace H in the simplest way, i.e. we put $h_k^i = 0$. Thus, H is generated by the coordinate vector field $\partial/\partial y$. Then the general formulae of the foregoing section lead (with some easy computations for the adjoints) to the following expressions:

	f	$f\tau$	$f\nu$	$f\tau \otimes \nu$
d_L	$\frac{\partial f}{\partial x}\tau$	0	$\frac{\partial f}{\partial x}\tau \otimes \nu$	0
d_{II}	$\frac{\partial f}{\partial y}\nu$	$-\frac{\partial f}{\partial y}\tau \otimes \nu$	0	0
d_L^*	0	$-\frac{\partial f}{\partial x}$	0	$-\frac{\partial f}{\partial x}\nu$
d_{II}^*	0	0	$-\frac{\partial f}{\partial y}$	$\frac{\partial f}{\partial y}\tau$

,

with $f \in C^\infty(\mathbb{T}^2)$. To prove, e.g.,

$$\mathrm{d}_\mathrm{H}^*(f\tau \otimes \nu) = \frac{\partial f}{\partial y}\tau,$$

we denote by $(\cdot|\cdot)$ the scalar product in $L^2(\mathbb{T}^2, E)$ and observe that

$$\begin{aligned}(g\tau|\mathrm{d}_\mathrm{H}^*(f\tau \otimes \nu)) &\equiv (\mathrm{d}_\mathrm{H}(g\tau)|f\tau \otimes \nu) = \left(-\frac{\partial g}{\partial y}\tau \otimes \nu | f\tau \otimes \nu\right)\\ &= -\int \frac{\partial g(x,y)}{\partial y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int g(x,y)\frac{\partial f(x,y)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y\\ &= \left(g\tau \,\middle|\, \frac{\partial f}{\partial y}\tau\right).\end{aligned}$$

Note that all the above operators can also be written as matrix differential operators.

3.2. The first order signature operator as Dirac operator

We will first show that $(C^\infty(\mathbb{T}^2) \rtimes \mathbb{R}, L^2(\mathbb{T}^2, E), \tilde{Q})$, with $\tilde{Q}$ being the linear signature operator

$$\tilde{Q} = \mathrm{d}_\mathrm{L} + \mathrm{d}_\mathrm{L}^* + \mathrm{d}_\mathrm{H} + \mathrm{d}_\mathrm{H}^*,$$

is a spectral triple of dimension 2. Using the foliation chart (x, y) and the local frame $\{\mathbf{1}, \tau, \nu, \tau \otimes \nu\}$, this operator can be written as

$$\tilde{Q} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}.$$

Its eigenvalues are given by

$$\lambda_{kl}^\pm = \pm\sqrt{(ak+bl)^2 + (al-bk)^2} \tag{3.9}$$

with $k, l \in \mathbb{Z}$ and it is straightforward to see that

$$\begin{aligned} e_{kl}^{+1} &= -\mathrm{i}\frac{bk-al}{\lambda_{kl}^+}e_{kl}^1 + e_{kl}^3 + \mathrm{i}\frac{ak+bl}{\lambda_{kl}^+}e_{kl}^4, \quad & e_{kl}^{+2} &= -\mathrm{i}\frac{ak+bl}{\lambda_{kl}^+}e_{kl}^1 + e_{kl}^2 + \mathrm{i}\frac{al-bk}{\lambda_{kl}^+}e_{kl}^4,\\ e_{kl}^{-1} &= e_{kl}^1 - \mathrm{i}\frac{ak+bl}{\lambda_{kl}^+}e_{kl}^2 + \mathrm{i}\frac{al-bk}{\lambda_{kl}^+}e_{kl}^3, \quad & e_{kl}^{-2} &= \mathrm{i}\frac{al-bk}{\lambda_{kl}^+}e_{kl}^2 + \mathrm{i}\frac{ak+bl}{\lambda_{kl}^+}e_{kl}^3 + e_{kl}^4, \end{aligned}$$

with

$$e^1_{kl}=\begin{pmatrix}e_{kl}\\0\\0\\0\end{pmatrix},\qquad e^2_{kl}=\begin{pmatrix}0\\e_{kl}\\0\\0\end{pmatrix},\qquad e^3_{kl}=\begin{pmatrix}0\\0\\e_{kl}\\0\end{pmatrix},\qquad e^4_{kl}=\begin{pmatrix}0\\0\\0\\e_{kl}\end{pmatrix},$$

form a complete set of eigenvectors.

Proposition 3. *$(C^\infty(\mathbb{T}^2)\rtimes\mathbb{R}, L^2(\mathbb{T}^2,E),\tilde{Q})$ is a spectral triple of dimension* 2.

Proof. The eigenvalues (3.9) of $\tilde{Q}$ have finite multiplicity, tend to infinity for $k,l\to\infty$ and have no finite accumulation point. Thus, the resolvent of $\tilde{Q}$ is compact. Since

$$[\tilde{Q},V_t]=0, \tag{3.10}$$

boundedness of the commutators of $\tilde{Q}$ with elements of the algebra follows from the fact that

$$[\tilde{Q},fV_t]=f[\tilde{Q},V_t]+[\tilde{Q},f]V_t=\begin{pmatrix}0 & -\dfrac{\partial f}{\partial x} & -\dfrac{\partial f}{\partial y} & 0\\ \dfrac{\partial f}{\partial x} & 0 & 0 & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial f}{\partial y} & 0 & 0 & -\dfrac{\partial f}{\partial x}\\ 0 & -\dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial x} & 0\end{pmatrix}V_t$$

is a bounded matrix multiplication operator in $L^2(\mathbb{T}^2,E)$. In order to see that the nth eigenvalue of $|\tilde{Q}|$ is of order $\sqrt{n}$ notice first that the eigenvalues of $|\tilde{Q}|$ are λ^+_{kl}, with multiplicity $4\times$ (number of $(k,l)\in\mathbb{Z}^2$ leading to the same λ^+_{kl}). The number of eigenvalues with absolute value less than some $R>0$ is then $4\times$ (number of integer lattice points inside a circle of radius R), i.e. equal to $4\times$ (the area πR^2) up to lower order terms in R. (Recall that $(x,y)\mapsto(\vartheta_1,\vartheta_2)$ is orthogonal.) This proves the claim. □

In order to describe the differential algebra $\Omega_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2)\rtimes\mathbb{R})$, we denote, as in formulae (2.1) and (2.2), by π^1 and π^2 the extensions of π to universal one and two forms. Since π is faithful by Proposition 2, $\Omega^1_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2)\rtimes\mathbb{R})$ is isomorphic to $\pi^1(\Omega^1(\mathcal{O}(\mathbb{T}^2)\rtimes\mathbb{R}))$, with $\mathrm{d}u_j\mapsto[\tilde{Q},U_j]$, $\mathrm{d}v_t\mapsto[\tilde{Q},V_t]$, and $\Omega^2_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2)\rtimes\mathbb{R})=\Omega^2(\mathcal{O}(\mathbb{T}^2)\rtimes\mathbb{R})/(\ker\pi^2+\mathrm{d}(\ker\pi^1))\simeq\pi^2(\Omega^2(\mathcal{O}(\mathbb{T}^2)\rtimes\mathbb{R}))/\pi^2(\mathrm{d}(\ker\pi^1))$.

Let us first note that, under the obvious identification $L^2(\mathbb{T}^2,E)\simeq\mathbb{C}^4\otimes L^2(\mathbb{T}^2)$ the generators U_1,U_2,V_t of $\mathcal{O}(\mathbb{T}^2)$ and its commutators with $\tilde{Q}$ can be written as follows:

$$U_1=\mathbf{1}\otimes s_1,\qquad U_2=\mathbf{1}\otimes s_2,\qquad V_t=\mathbf{1}\otimes v_{abt}, \tag{3.11}$$

where $s_1 e_{kl} = e_{k+1,l}$, $s_2 e_{kl} = e_{k,l+1}$, $v_{abt} e_{kl} = \mathrm{e}^{\mathrm{i}(ak+bl)t} e_{kl}$, and

$$[\tilde{Q}, U_1] = \begin{pmatrix} 0 & a & -b & 0 \\ -a & 0 & 0 & b \\ b & 0 & 0 & a \\ 0 & -b & -a & 0 \end{pmatrix} \otimes s_1, \quad [\tilde{Q}, U_2] = \begin{pmatrix} 0 & b & a & 0 \\ -b & 0 & 0 & -a \\ -a & 0 & 0 & b \\ 0 & a & -b & 0 \end{pmatrix} \otimes s_2. \tag{3.12}$$

Using this representation, together with $[s_1, s_2] = 0$, $s_1 v_{abt} = \mathrm{e}^{\mathrm{i}at} v_{abt} s_1$, $s_2 v_{abt} = \mathrm{e}^{\mathrm{i}bt} v_{abt} s_2$, it is easy to show the following lemma.

Lemma 2.

$$U_j[\tilde{Q}, U_k] = [\tilde{Q}, U_k]U_j \quad \forall j, k \in \{1, 2\}, \tag{3.13}$$

$$V_t[\tilde{Q}, U_1] = \mathrm{e}^{\mathrm{i}at}[\tilde{Q}, U_1]V_t, \qquad V_t[\tilde{Q}, U_2] = \mathrm{e}^{\mathrm{i}bt}[\tilde{Q}, U_2]V_t, \tag{3.14}$$

$$[\tilde{Q}, U_1][\tilde{Q}, U_2] = -[\tilde{Q}, U_2][\tilde{Q}, U_1]. \tag{3.15}$$

Explicitly, we have

$$[\tilde{Q}, U_1][\tilde{Q}, U_2] = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \otimes s_1 s_2. \tag{3.16}$$

Proposition 4.

(i) $\Omega^1_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R})$ *is a free left (and right)* $\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$*-module with basis* $\{\mathrm{d}u_1, \mathrm{d}u_2\}$. *Its bimodule structure is determined by*

$$u_j\, \mathrm{d}u_k = \mathrm{d}u_k\, u_j \quad \forall j, k \in \{1, 2\}, \tag{3.17}$$

$$v_t\, \mathrm{d}u_1 = \mathrm{e}^{\mathrm{i}at}\, \mathrm{d}u_1 v_t, \qquad v_t\, \mathrm{d}u_2 = \mathrm{e}^{\mathrm{i}bt}\, \mathrm{d}u_2 v_t. \tag{3.18}$$

Moreover,

$$\mathrm{d}v_t = 0. \tag{3.19}$$

(ii) $\Omega^2_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R})$ *is a free left (and right)* $\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$*-module with basis* $\{\mathrm{d}u_1\, \mathrm{d}u_2\}$, *with*

$$\mathrm{d}u_1\, \mathrm{d}u_2 = -\mathrm{d}u_2\, \mathrm{d}u_1. \tag{3.20}$$

(iii) $\Omega^k_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}) = 0$ *for* $k \geq 3$.

Proof. For the proof we refer to [26]. □

Remark 4. One can define a first order differential calculus for the algebra $\mathcal{A} = C^\infty(\mathbb{T}^2) \rtimes \mathbb{R}$ in the following way. Let $\Omega^1_{\tilde{Q}}(\mathcal{A})$ be the free left $\mathcal{A}$-module with basis $\{\mathrm{d}u_1, \mathrm{d}u_2\}$. Equipped with the product of the topologies defined by the sequence of seminorms $p_n^{\rtimes}$, $\Omega^1_{\tilde{Q}}(\mathcal{A})$ is a free left topological $\mathcal{A}$-module. One turns $\Omega^1_{\tilde{Q}}(\mathcal{A})$ into a right $\mathcal{A}$-module by defining $a_j\,\mathrm{d}u_j\,u_k := a_j u_k\,\mathrm{d}u_j$ for $j, k \in 1, 2$, $a_1\,\mathrm{d}u_1\,v_t := \mathrm{e}^{\mathrm{i}at}a_1 v_t\,\mathrm{d}u_1$ and $a_2\,\mathrm{d}u_2\,v_t := \mathrm{e}^{\mathrm{i}bt}a_2 v_t\,\mathrm{d}u_2$ ($a_j \in \mathcal{A}$), and extending this by continuity. This gives $\Omega^1_{\tilde{Q}}(\mathcal{A})$ the structure of a topological bimodule containing $\Omega^1_{\tilde{Q}}(\mathcal{O})$ as a dense subbimodule. It is also not difficult to see that the differential can be extended to a continuous map $\mathrm{d} : \mathcal{A} \to \Omega^1_{\tilde{Q}}(\mathcal{A})$. Analogously, one can define a topological $\mathcal{A}$-bimodule $\Omega^1_{\tilde{Q}}(\mathcal{A})$ such that the natural mappings $\Omega^1_{\tilde{Q}}(\mathcal{A}) \times \Omega^1_{\tilde{Q}}(\mathcal{A}) \to \Omega^2_{\tilde{Q}}(\mathcal{A})$ and $\mathrm{d} : \Omega^1_{\tilde{Q}} \to \Omega^2_{\tilde{Q}}(\mathcal{A})$ are continuous. We conjecture that the differential calculus $\Omega_{\tilde{Q}}(\mathcal{A})$ so constructed coincides with the calculus (to be denoted by the same symbol) resulting from the spectral triple $(\mathcal{A}, L^2(\mathbb{T}^2, E), \tilde{Q})$.

3.3. *The mixed signature operator*

Let us now consider the mixed signature operator Q given by formula (2.5). In matrix representation, we have

$$Q = \begin{pmatrix} \dfrac{\partial^2}{\partial x^2} & 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & -\dfrac{\partial^2}{\partial x^2} & 0 & -\dfrac{\partial}{\partial y} \\ -\dfrac{\partial}{\partial y} & 0 & -\dfrac{\partial^2}{\partial x^2} & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 & \dfrac{\partial^2}{\partial x^2} \end{pmatrix}.$$

In order to diagonalize this operator, we have to solve the eigenvalue problem

$$Q \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \tag{3.21}$$

with $f_i \in L^2(\mathbb{T}^2, E)$. The operator Q is already block-diagonal and acts both in the space of (0, 0)- and (0, 1)-forms and in the space of (1, 1)- and (1, 0)-forms in the same way. It suffices to diagonalize one block. With $g = f_1 + f_3$ and $h = f_1 - f_3$ one arrives at

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial y} = \lambda g, \qquad \frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial y} = \lambda h,$$

and the ansatz

$$g = \sum_{k,l\in\mathbb{Z}} \eta_{kl}\, \mathrm{e}^{\mathrm{i}(k\vartheta_1+l\vartheta_2)}, \qquad h = \sum_{k,l\in\mathbb{Z}} \chi_{kl}\, \mathrm{e}^{\mathrm{i}(k\vartheta_1+l\vartheta_2)}$$

leads to

$$((a^2k^2 + 2abkl + b^2l^2)^2 + (bk - al)^2)\chi_{kl} = \lambda^2\chi_{kl},$$

which gives the eigenvalues

$$\lambda_{kl\pm} = \pm\sqrt{(ak+bl)^4 + (bk-al)^2}.$$

One easily concludes that eigenvectors to the eigenvalues $\lambda_{kl\pm}$ are of the form

$$h_{kl} = e_{kl}, \qquad g_{kl\pm} = \gamma_{kl\pm}e_{kl}$$

with

$$\gamma_{kl\pm} = \frac{-(ak+bl)^2 + \mathrm{i}(al-bk)}{\lambda_{kl\pm}}.$$

The eigenvectors of the original problem (3.21) are

$$f_{1_{kl\pm}} = \tfrac{1}{2}(g_{kl\pm} + h_{kl}) = \tfrac{1}{2}(1+\gamma_{kl\pm})e_{kl}, \quad f_{3_{kl\pm}} = \tfrac{1}{2}(g_{kl\pm} - h_{kl}) = \tfrac{1}{2}(\gamma_{kl\pm} - 1)e_{kl},$$

or, written as elements of $L^2(\mathbb{T}^2, E)$,

$$e^{(1)}_{kl\pm} = \tfrac{1}{2}e_{kl}((\gamma_{kl\pm}+1)\mathbf{1} + (\gamma_{kl\pm}-1)\nu).$$

If we assume that the metrics are chosen so that the frame elements $\mathbf{1}, \tau, \nu, \tau\otimes\nu$ are orthonormal, these vectors are already orthonormal (note that $|\gamma_{kl\pm}| = 1$). The same argument yields another set

$$e^{(2)}_{kl\pm} = \tfrac{1}{2}e_{kl}((\gamma_{kl\pm}+1)\tau\otimes\nu + (\gamma_{kl\pm}-1)\tau)$$

of eigenvectors to the same eigenvalues $\lambda_{kl\pm}$. Note that the eigenvalue 0 appears only for $k = l = 0$. In that case, the eigenvalue equations decouple, and we get four independent eigenvectors $\mathbf{1}, \tau, \nu, \tau\otimes\nu$. In order to see that these vectors together with the $e^{(1,2)}_{kl\pm}$ form an orthonormal basis of $L^2(\mathbb{T}^2, E)$, it is sufficient to see that all the vectors $e_{kl}\mathbf{1}, e_{kl}\tau, e_{kl}\nu, e_{kl}\tau\otimes\nu$ are linear combinations of the foregoing vectors. This follows from the fact that the matrix

$$\begin{pmatrix} \gamma_{kl+}+1 & \gamma_{kl+}-1 \\ \gamma_{kl-}+1 & \gamma_{kl-}-1 \end{pmatrix}$$

is always invertible (its determinant being $-4\gamma_{kl+}$).

Thus, we have found the spectral decomposition of the selfadjoint operator Q. Its unboundedness is reflected in the unboundedness of the $\lambda_{kl\pm}$. It is now easy to write down also the spectral decomposition of the corresponding Dirac operator D: applying (2.7) for nonzero eigenvalues gives

$$De^{(1,2)}_{kl\pm} = \pm\sqrt{\lambda_{kl}}e^{(1,2)}_{kl\pm},$$

where λ_{kl} is the positive root λ_{kl+}. Putting $e^{(1)}_{00+} = \mathbf{1}$, $e^{(1)}_{00-} = \nu$, $e^{(2)}_{00+} = \tau\otimes\nu$ and $e^{(2)}_{00-} = \tau$, the formula defines D also on the kernel of Q (cf. (2.6)), and gives the spectral decomposition of D.

Defining

$$\eta^{(1,2)}_{kl\pm} := \tfrac{1}{2}(e^{(1,2)}_{kl+} \pm e^{(1,2)}_{kl-}),$$

one finds

$$U_1\eta^{(1,2)}_{kl+} = \eta^{(1,2)}_{k+1,l+}, \tag{3.22}$$

$$U_1\eta^{(1,2)}_{kl-} = \frac{\gamma_{kl}}{\gamma_{k+1,l}}\eta^{(1,2)}_{k+1,l-}, \tag{3.23}$$

$$U_2\eta^{(1,2)}_{kl+} = \eta^{(1,2)}_{k,l+1,+}, \tag{3.24}$$

$$U_2\eta^{(1,2)}_{kl-} = \frac{\gamma_{kl}}{\gamma_{k,l+1}}\eta^{(1,2)}_{k,l+1,-}, \tag{3.25}$$

$$V_t\eta^{(1,2)}_{kl\pm} = \mathrm{e}^{\mathrm{i}(ka+lb)t}\eta^{(1,2)}_{kl\pm}, \tag{3.26}$$

$$D\eta^{(1,2)}_{kl\pm} = \sqrt{\lambda_{kl}}\eta^{(1,2)}_{kl\mp}. \tag{3.27}$$

From Theorem 1 or by direct computation using (3.22)–(3.27) one gets the following proposition.

Proposition 5. *$(C^\infty(\mathbb{T}^2)\rtimes\mathbb{R}, L^2(\mathbb{T}^2, E), D)$ is a spectral triple of dimension* 3.

Next, one would like to describe the differential calculus Ω_D related to this spectral triple. Let us first show that the first order calculus for the restriction of the spectral triple to the subalgebra $C^\infty(\mathbb{T}^2)$ is the universal one. To begin with, we have the following lemma.

Lemma 3. *Let $p, q, r, s \in \mathbb{Z}$. Then we have*

$$U_1^rU_2^p[D, U_1^sU_2^q]\eta^{(1,2)}_{kl\pm} = \frac{\sqrt{\lambda_{k+s,l+q}}\gamma_{k+s,l+q} - \sqrt{\lambda_{kl}}\gamma_{kl}}{\gamma_{k+r+s,l+p+q}}\eta^{(1,2)}_{k+r+s,l+p+q\mp}, \quad [D, V_t] = 0.$$

Moreover,

$$V_t[D, U_1] = \mathrm{e}^{\mathrm{i}at}[D, U_1]V_t, \qquad V_t[D, U_2] = \mathrm{e}^{\mathrm{i}bt}[D, U_2]V_t. \tag{3.28}$$

Proof. By direct computation using (3.22)–(3.27). □

From Theorem 1 we know that the particular choice $\Gamma = \mathbf{1}$ gives rise to a spectral triple over $C^\infty(\mathbb{T}^2)$. Let us now first investigate the corresponding differential calculus

$\Omega_D(\mathcal{O}(\mathbb{T}^2))$. By faithfulness of the representation, we can again identify $\Omega^1_D(\mathcal{O}(\mathbb{T}^2))$ with a subspace of $B(L^2(\mathcal{H}))$. We have the following proposition.

Proposition 6. *The first order differential calculus $\Omega^1_D(\mathcal{O}(\mathbb{T}^2))$ is freely generated as a left $\mathcal{O}(\mathbb{T}^2)$-module by the elements $[D, U_1^s U_2^q]$ $(s, q \in \mathbb{Z})$.*

Proof. We show that no nontrivial relations between $U_1^s U_2^q$ and commutators $[D, U_1^t U_2^r]$ exist. Let us first consider relations involving D and U_1 only. From Lemma 3 it follows for $p = q = 0$ that

$$U_1^r[D, U_1^s]\eta^{(1,2)}_{kl\pm} = \frac{\sqrt{\lambda_{k+s,l}}\gamma_{k+s,l} - \sqrt{\lambda_{kl}}\gamma_{kl}}{\gamma_{k+r+s,l}}\eta^{(1,2)}_{k+r+s,l\mp}.$$

Using the Leibniz rule and the fact that different overall powers of U_1 are independent we find that nontrivial relations would be of the form

$$\sum_{m=0}^{s-1} a_m U_1^m [D, U_1^{s-m}] = 0 \tag{3.29}$$

for $s \in \mathbb{N}$. Applying (3.29) to $\eta^{(1,2)}_{n0\pm}$ $(n = k, \dots, k + s - 1)$, we get the following system of equations:

$$\begin{aligned}
&\textstyle\sum_{j=0}^{s-1} a_j(\sqrt{\lambda_{k+j+1,0}}\gamma_{k+j+1,0} - \sqrt{\lambda_{k0}}\gamma_{k0}) = 0\\
&\vdots\\
&\textstyle\sum_{j=0}^{s-1} a_j(\sqrt{\lambda_{k+j+s,0}}\gamma_{k+j+s,0} - \sqrt{\lambda_{k+s-1,0}}\gamma_{k+s-1,0}) = 0.
\end{aligned}$$

For the discussion of this system of equations it is useful to define a function h on $\mathbb{Z}$ putting

$$h(i) = \sqrt{\lambda_{i0}}\gamma_{i0} - \sqrt{\lambda_{i-1,0}}\gamma_{i-1,0}.$$

Lemma 4. *We have*

$$\begin{vmatrix} h(i_0) & \cdots & h(i_0 + k) \\ \vdots & \ddots & \vdots \\ h(i_k) & \cdots & h(i_k + k) \end{vmatrix} \neq 0$$

for all $k \in \mathbb{N}$ and $i_0, \dots, i_k \in \mathbb{Z}$.

Proof. See Appendix A. □

Thus, there are no relations involving U_1 and D besides the ones coming from the Leibniz rule. In the general case we are looking for $a_{mn} \in \mathbb{C}$ such that

$$\sum_{m=0}^{s-1}\sum_{n=0}^{q-1} a_{mn} U_1^m U_2^n [D, U_1^{s-m} U_2^{q-n}] = 0.$$

Again, we are led to the consideration of a homogeneous linear system of equations for the a_{mn}. The corresponding matrix of coefficients is an $(sq \times sq)$-matrix with general matrix element

$$C_{k,(m,n)} = (\sqrt{\lambda_{k+s-m,q-n}}\gamma_{k+s-m,q-n} - \sqrt{\lambda_{k0}}\gamma_{k0})$$

$(k = 1, \dots, sq)$. In analogy to the case discussed above we have the following lemma.

Lemma 5. *Let $s, q \in \mathbb{N}$ be fixed. Then we have*

$$\det(C_{k,(m,n)}) \neq 0.$$

Proof. The proof is a straightforward generalization of the proof of Lemma 4 to the case of functions defined on $\mathbb{Z}^2$, see [29]. □

The proof of the proposition follows now immediately from the fact that between the elements $[D, U_1^s U_2^q]$ there are no relations besides the ones coming from the Leibniz rule.

Unfortunately, we were not able to derive more relations of the type (3.28) between commutators of D with some generator and other generators (up to such relations resulting from applying $[D, \cdot]$ to (3.4)–(3.7) and the unitarity condition, using the derivation property). The difficulty comes from the fact that λ_{kl} and γ_{kl} contain both second and fourth powers of k and l under the square root, the latter stemming from the quadratic part of the mixed signature operator Q. This leads us to the following conjecture.

Conjecture 1. *The bimodule $\Omega^1_D(C^\infty(\mathbb{T}^2) \rtimes \mathbb{R})$ is generated by $\mathrm{d}u_1$ and $\mathrm{d}u_2$ and is described by two relations*

$$v_t\, \mathrm{d}u_1 = \mathrm{e}^{\mathrm{i}at}\, \mathrm{d}u_1 v_t, \qquad v_t\, \mathrm{d}u_2 = \mathrm{e}^{\mathrm{i}bt}\, \mathrm{d}u_2 v_t.$$

Let us note that we could choose another diffeomorphism group, restricting the action of $\mathbb{R}$ to the subgroup $\mathbb{Z}$. Then, the generators V_t (or v_t) would be reduced to one generator $V_1 = V$ $(v_1 = v)$, and all the above formulae remain, replacing always V_t (v_t) by some power of V (v). However, we would not get rid of the difficulties related to the differential calculus.

Let us now turn to the Chern character $\mathrm{ch}_*(\mathcal{A}, \mathcal{H}, D)$ of the spectral triple. We first obtain from (3.27) that the operator $F = D/|D|$ acts in the basis $\eta^{(1,2)}_{kl\pm}$ as follows:

$$F\eta^{(1,2)}_{kl\pm} = \eta^{(1,2)}_{kl\mp}. \tag{3.30}$$

Therefore, we find

$$[F, V_t] = 0, \qquad U_1^r U_2^p [F, U_1^s U_2^q]\eta^{(1,2)}_{kl\pm} = \frac{\gamma_{k+s,l+q} - \gamma_{kl}}{\gamma_{k+r+s,l+p+q}}\eta^{(1,2)}_{k+r+s,l+p+q\mp}. \tag{3.31}$$

Since the spectral triple is 3-summable we get (cf. (2.9))

$$\mathrm{ch}_*(\mathcal{A}, \mathcal{H}, D)(a^0, a^1, a^2, a^3) = \lambda_3 \operatorname{Tr}(a^0[F, a^1][F, a^2][F, a^3]), \tag{3.32}$$

for all $a^i \in C^\infty(\mathbb{T}^2) \rtimes \mathbb{R}$, with $\lambda_3 = -(3/4)\sqrt{2\pi \mathrm{i}}$. The above commutators (3.32) act as weighted shift operators and therefore we have

$$\mathrm{ch}_*(\mathcal{A}, \mathcal{H}, D) = 0. \tag{3.33}$$

We note that the explicit form of the spectral triple, in particular the construction of the eigenvalue basis for the Dirac operator, makes it possible to calculate the right-hand side of the local index formula of Connes and Moscovici [16] giving also zero for the Chern character.

In order to calculate the Dixmier trace of the Dirac operator D we have to find the homogeneous part of order -3 of its symbol (in the calculus of hypoelliptic operators $\psi DO'$). It is easy to see (using [16, Formula (26)]) that

$$\sigma(D^{2k})(\xi_v,\xi_n)=(\xi_v^4+\xi_n^2)^{k/2}\mathbf{1}_4,$$
$$\sigma(D^{2k+1})(\xi_v,\xi_n)=(\xi_v^4+\xi_n^2)^{(2k-1)/4}\begin{pmatrix}-\xi_v^2 & 0 & \mathrm{i}\xi_n & 0\\ 0 & \xi_v^2 & 0 & -\mathrm{i}\xi_n\\ -\mathrm{i}\xi_n & 0 & \xi_v^2 & 0\\ 0 & \mathrm{i}\xi_n & 0 & -\xi_v^2\end{pmatrix} \tag{3.34}$$

for all $k\in\mathbb{Z}$. For the degree of homogeneity one finds

$$\sigma(D^m)(\lambda\xi_v,\lambda^2\xi_n)=\lambda^m\sigma(D^m)(\xi_v,\xi_n) \tag{3.35}$$

for $m\in\mathbb{Z}$, $\lambda\in\mathbb{R}_+^*$. Therefore, we have $\sigma_{-3}(D^k)=0$ for all integers $k\neq-3$ and from (2.12) and (2.13) we find

$$\mathrm{Tr}_\omega(D^k)=0 \tag{3.36}$$

for all $k\in\mathbb{Z}$, $k\neq-3$. Since $\mathrm{tr}\,\sigma_{-3}(D^{-3})(x,\xi)=0$, see (3.34), we also get

$$\mathrm{Tr}_\omega(D^{-3})=0.$$

We remark that $|D|^{-3}$ has the nonvanishing Dixmier trace, cf. [16, Formula (66)]

$$\mathrm{Tr}_\omega(|D|^{-3})=\frac{8}{3\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^2. \tag{3.37}$$

4. Discussion

With the obtained results one can try and discuss simple noncommutative field theoretical models. Unfortunately, in view of the results on the differential calculus, Proposition 6 and Conjecture 1, a gauge theory would contain an infinite tower of fields. For the spectral triple with the linear signature operator $\tilde{Q}$ as Dirac operator, which has a nice differential calculus, one would expect to get gauge theories similarly as in [27]. On the other hand, the Dixmier trace can be used to construct a candidate for a gravity action by considering $\mathrm{Tr}_\omega(D^{2-d})$, where d is the dimension of the spectral triple, see [4,5]. However, for our three-dimensional triple we find from (3.36)

$$\mathrm{Tr}_\omega(D^{2-3})=\mathrm{Tr}_\omega(D^{-1})=0. \tag{4.1}$$

The above discussion indicates that the procedure of Connes and Moscovici [16] makes it possible to construct and discuss spectral triples for algebras related to foliations of smooth manifolds quite explicitly. Of course, the discussion of the Kronecker foliation can be only a first step towards a detailed understanding of this important class of algebras. We remark that this foliation is completely missing the essential feature of holonomy. It will be interesting to see whether and to what extent the occurrence of holonomy (e.g. for the Reeb foliation) will change the properties of the spectral triples.

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Appendix A. Proof of Lemma 4

The proof of this lemma rests on the following characterization of functions f defined by determinants of Hankelian type, see [28], such that

$$\begin{vmatrix} f(i_0) & \cdots & f(i_0+k) \\ \vdots & \ddots & \vdots \\ f(i_k) & \cdots & f(i_k+k) \end{vmatrix} = 0 \tag{A.1}$$

$\forall k \in \mathbb{N}$ and $i_0, \ldots, i_k \in \mathbb{Z}$. We have the following theorem.

Theorem A.1. *A function f defined on $\mathbb{Z}$ fulfils (A.1) if and only if it is of one of the following two types*:

$$f_1(i) = \beta^i \sum_{j=0}^{k-1} \alpha_j i^j, \tag{A.2}$$

$$f_2(i) = \sum_{j=1}^{k} \alpha_j \beta_j^i \tag{A.3}$$

with α, β and $\beta_j \in \mathbb{C}$.

Proof. By induction one easily shows that f_1 and f_2 fulfil (A.1).

Let us now assume that a function f defined on $\mathbb{Z}$ fulfils (A.1) for some $k \in \mathbb{N}$. We choose $i_1 = i_0 + 1, \ldots, i_k = i_0 + k$ and let $f(i_0), \ldots, f(i_0 + 2k - 1)$ denote the corresponding values of f. Then $f(i_0 + 2k)$ has to fulfil

$$\begin{vmatrix} f(i_0) & f(i_0+1) & \cdots & f(i_0+k) \\ f(i_0+1) & f(i_0+2) & \cdots & f(i_0+k+1) \\ \vdots & \vdots & \ddots & \vdots \\ f(i_0+k) & f(i_0+1) & \cdots & f(i_0+2k) \end{vmatrix} = 0,$$

provided that

$$\begin{vmatrix} f(i_0) & f(i_0+1) & \cdots & f(i_0+k-1) \\ f(i_0+1) & f(i_0+2) & \cdots & f(i_0+k) \\ \vdots & \vdots & \ddots & \vdots \\ f(i_0+k-1) & f(i_0+k) & \cdots & f(i_0+2k-2) \end{vmatrix} \neq 0. \tag{A.4}$$

(We may assume without loss of generality that (A.4) holds. In [29], it is shown that in the other case one is led to the case $k-1$.) Proceeding further we find that the $2k$ values $f(i_0), \dots, f(i_0+2k-1)$ determine f completely. Now we show that this function is either of type (A.3) or (A.2).

Let first the constants $f(i_0), \dots, f(i_0+2k-1)$ be such that the following condition holds:

$$\sum_{j=0}^{l+1} \beta^{l+1-j} f(i+j)(-1)^j \binom{l+1}{j} = 0 \tag{A.5}$$

for some $\beta \in \mathbb{C}$, $l \in \{0, \dots, k-1\}$ and all $i = i_0, \dots, i_0+2k-l-1$. We show that the corresponding function on $\mathbb{Z}$ is of the form (A.2). Suppose that $\beta \in \mathbb{C}$ is a solution of (A.5). Then we find constants α_i as follows. Defining $g(i) := f(i)/\beta^i$ ($\beta \neq 0$), we can always find α_i ($i = 0, \dots, l$) as solutions of the following linear system of equations:

$$\begin{aligned} &g(i_0) = \alpha_0 + \alpha_1 i_0 + \cdots + \alpha_l i_0^l, \\ &g(i_0+1) = \alpha_0 + \alpha_1(i_0+1) + \cdots + \alpha_l(i_0+1)^l \\ &\vdots \\ &g(i_0+l) = \alpha_0 + \alpha_1(i_0+l) + \cdots + \alpha_l(i_0+l)^l \end{aligned}$$

by

$$\alpha_i = \frac{1}{\Delta} \begin{vmatrix} 1 & i_0 & \cdots & i_0^{i-1} & g(i_0) & i_0^{i+1} & \cdots & i_0^l \\ 1 & i_0+1 & \cdots & (i_0+1)^{i-1} & g(i_0+1) & (i_0+1)^{i+1} & \cdots & (i_0+1)^l \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & i_0+l & \cdots & (i_0+l)^{i-1} & g(i_0+l) & (i_0+l)^{i+1} & \cdots & (i_0+l)^l \end{vmatrix}$$

with

$$\Delta = \begin{vmatrix} 1 & i_0 & \cdots & i_0^l \\ 1 & i_0+1 & \cdots & (i_0+1)^l \\ \vdots & \vdots & \ddots & \vdots \\ 1 & i_0+l & \cdots & (i_0+l)^l \end{vmatrix} = (-1)^{l(l+1)/2} \prod_{j=1}^{l} j! \neq 0.$$

Now that we have chosen the constants β and $\alpha_0, \dots, \alpha_l$ such that

$$f(i_0+j) = f_1(i_0+j)$$

is fulfilled, for all $j = 0, \dots, l$, it remains to be shown that we also have

$$f(i_0 + l + 1) = f_1(i_0 + l + 1).$$

But

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} j^s = 0$$

$\forall s = 0, \dots, r-1$ (which follows from evaluating the sth derivative of $f(x) = (x-1)^r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j x^{r-j}$ at $x = 1$). Now we find

$$\begin{aligned}
&\sum_{j=0}^{l+1} \beta^{l+1-j} (-1)^j \binom{l+1}{j} f_1(i+j) \\
&\quad = \sum_{j=0}^{l+1} \beta^{l+1-j} \beta^{i+j} (-1)^j \binom{l+1}{j} \sum_{m=0}^{l} \alpha_m (i+j)^m \\
&\quad = \beta^{l+i+1} \sum_{j=0}^{l+1} \sum_{m=0}^{l} \sum_{n=0}^{m} \alpha_m (-1)^j \binom{l+1}{j} \binom{m}{n} i^n j^{m-n} \\
&\quad = \beta^{l+i+1} \sum_{m=0}^{l} \alpha_m \sum_{n=0}^{m} \binom{m}{n} i^n \sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} j^{m-n} = 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{j=0}^{l+1} (-1)^j \beta^{l+1-j} \binom{l+1}{j} (f(i_0+j) - f_1(i_0+j)) \\
&\quad = (-1)^{l+1} (f(i_0+l+1) - f_1(i_0+l+1)) = 0,
\end{aligned}$$

i.e.

$$f(i_0 + l + 1) = f_1(i_0 + l + 1).$$

Let us now consider the general case (A.3). Suppose that $f(i_0), \dots, f(i_0 + 2k - 1)$ are chosen such that (A.5) does not hold. Then we have to solve the following system of algebraic equations (where we have chosen $i_0 = 0$):

$$\begin{aligned}
f(0) &= C_1 + \cdots + C_k \\
f(1) &= C_1\beta_1 + \cdots + C_k\beta_k \\
&\vdots \\
f(2k-1) &= C_1\beta_1^{2k-1} + \cdots + C_k\beta_k^{2k-1},
\end{aligned}$$

which can always be done using Gröbner basis techniques, see [29]. □

Remark A.1. If the parameters $f(i_0), \dots, f(i_0+2k-1)$ satisfy

$$f(i_0+l) = \frac{f(i_0+1)^l}{f(i_0)^{l-1}}$$

$\forall l = 0, \dots, 2k-1$, then one easily checks that

$$\beta = \frac{f(i_0+1)}{f(i_0)}$$

fulfils (A.5) and the constants α_i are given by

$$\alpha_0 = \frac{f(i_0)^{i_0+1}}{f(i_0+1)^{i_0}}, \qquad \alpha_1 = \cdots = \alpha_{k-1} = 0.$$

The proof of Lemma 4 follows now immediately from the observation that the function

$$h(i) = \sqrt{\lambda_{i0}}\gamma_{i0} - \sqrt{\lambda_{i-1,0}}\gamma_{i-1,0},$$

is obviously not of the form (A.2) or (A.3). □

Appendix B. The differential algebra for the irrational rotation algebra

Let us first recall, see [1,18], that the algebra of the noncommutative torus is generated by two unitaries u, v subject to the relation

$$uv = \mathrm{e}^{-2\pi \mathrm{i}\theta}\, vu.$$

The algebra can be considered on the purely $*$-algebraic level (Laurent polynomials in u, v) where a general element is a finite linear combination of ordered polynomials $u^k v^l, k, l \in \mathbb{Z}$, on the level of smooth functions, where the general element is a series $\sum a_{kl}u^k v^l$ with coefficients a_{kl} subject to the condition that $(|k|^n + |l|^n)|a_{kl}|$ are bounded for all $n > 0$. Finally, there is also the C^* version, defined, e.g. by using irreducible representations for performing a norm closure of the polynomial algebra. It is well known that both the polynomial and the C^*-algebra can be interpreted as convolution algebras of the reduced holonomy groupoid of the Kronecker foliation, which θ being the angle defining the direction of the leaves. We denote the C^*-algebra by A_θ, the smooth algebra by $\mathcal{A}_\theta$ and the polynomial algebra by $\mathcal{O}_\theta$. There exists a tracial state τ on A_θ, given by

$$\tau\left(\sum a_{kl}u^k v^l\right) = a_{00},$$

and there are two canonical derivations δ_1 and δ_2 on $\mathcal{A}_\theta$ defined by

$$\delta_1(u^k v^l) = 2\pi \mathrm{i} k u^k v^l, \qquad \delta_2(u^k v^l) = 2\pi \mathrm{i} l u^k v^l.$$

With these data, the well-known spectral triple is defined as follows. First, the tracial state τ is used to define the GNS Hilbert space $\mathcal{H}_\tau$. Secondly, the derivations δ_1 and δ_2 give rise to unbounded operators on $\mathcal{H}_\tau$, whose domain of definition is the image of $\mathcal{A}_\theta$ in $\mathcal{H}_\tau$ (under the GNS procedure). The same is true for $\partial := (1/\sqrt{2\pi})(\delta_1 - \mathrm{i}\delta_2)$. Now take

$\mathcal{H} := \mathcal{H}_\tau \oplus \mathcal{H}_\tau$ and

$$D := \begin{pmatrix} 0 & \partial \\ \partial^* & 0 \end{pmatrix}$$

as Hilbert space and generalized Dirac operator of a spectral triple over $\mathcal{A}_\theta$. The dimension of this spectral triple is known to be 2. The corresponding differential calculus Ω_D was described by Connes in terms of elements of $\mathcal{H}$. We have the following description of $\Omega_D(\mathcal{O}_\theta)$ in terms of relations between the generators of the algebra and their differentials.

Proposition B.1.

(i) *$\Omega^1_D(\mathcal{O}_\theta)$ is a free left (or right) $\mathcal{O}_\theta$-module with basis $\{\mathrm{d}u, \mathrm{d}v\}$. The bimodule structure of $\Omega^1_D(\mathcal{O}_\theta)$ is given by*

$$u\,\mathrm{d}u = \mathrm{d}u\,u, \qquad u^*\,\mathrm{d}u = \mathrm{d}u\,u^*, \qquad u\,\mathrm{d}u^* = \mathrm{d}u^*\,u, \qquad u^*\,\mathrm{d}u^* = \mathrm{d}u^*\,u^*, \tag{B.1}$$

$$v\,\mathrm{d}v = \mathrm{d}v\,v, \qquad v^*\,\mathrm{d}v = \mathrm{d}v\,v^*, \qquad v\,\mathrm{d}v^* = \mathrm{d}v^*\,v, \qquad v^*\,\mathrm{d}v^* = \mathrm{d}v^*\,v^*, \tag{B.2}$$

$$v\,\mathrm{d}u = \mathrm{e}^{2\pi \mathrm{i}\theta}\,\mathrm{d}u\,v, \qquad u\,\mathrm{d}v = \mathrm{e}^{-2\pi \mathrm{i}\theta}\,\mathrm{d}v, \tag{B.3}$$

$$v\,\mathrm{d}u^* = \mathrm{e}^{-2\pi \mathrm{i}\theta}\,\mathrm{d}u^*\,v, \qquad u^*\,\mathrm{d}v = \mathrm{e}^{2\pi \mathrm{i}\theta}\,\mathrm{d}v\,u^*, \tag{B.4}$$

$$v^*\,\mathrm{d}u = \mathrm{e}^{-2\pi \mathrm{i}\theta}\,\mathrm{d}u\,v^*, \qquad u\,\mathrm{d}v^* = \mathrm{e}^{2\pi \mathrm{i}\theta}\,\mathrm{d}v^* u. \tag{B.5}$$

(ii) *$\Omega^2_D(\mathcal{A}_\theta)$ is a free left (or right) $\mathcal{A}_\theta$-module with basis $\{\mathrm{d}u\,\mathrm{d}v\}$. The relation*

$$\mathrm{d}u\,\mathrm{d}v = -\mathrm{e}^{2\pi \mathrm{i}\theta}\,\mathrm{d}v\,\mathrm{d}u \tag{B.6}$$

is fulfilled.

(iii) *$\Omega^k_D(\mathcal{A}_\theta) = 0$ for $k \geq 3$.*

Proof.

(i) τ is a faithful state, thus the GNS representation π is faithful. Consequently, $\Omega^1_D(\mathcal{O}_\theta) \simeq \pi(\Omega^1(\mathcal{O}_\theta))$, where the isomorphism sends differentials to commutators with D. To verify the relations (B.1)–(B.5) it is therefore sufficient to consider the images of these expressions under π. If we denote by $\underline{a}$ the element corresponding to $a \in \mathcal{O}_\theta$ in $\mathcal{H}_\tau$, it is immediately verified that the $e_{kl} := \underline{u^k v^l}$ form an orthonormal basis in $\mathcal{H}_\tau$. From this basis we obtain in an obvious way an orthonormal basis $\{e^+_{kl}, e^-_{kl}\}$ of $\mathcal{H}_\tau \oplus \mathcal{H}_\tau$ (to be precise, $e^+_{kl} = (e_{kl}, 0)$, $e^-_{kl} = (0, e_{kl})$). In this basis, $U := \pi \oplus \pi(u)$, $V := \pi \oplus \pi(v)$, D act as follows:

$$U(e^\pm_{kl}) = e^\pm_{k+1,l}, \qquad V(e^\pm_{kl}) = \mathrm{e}^{2\pi \mathrm{i}k\theta} e^\pm_{k,l+1}, \tag{B.7}$$

$$D(e^\pm_{kl}) = \sqrt{2\pi}(\pm \mathrm{i}k + l)e^\mp_{kl}. \tag{B.8}$$

Now, it is straightforward to verify the relations (B.1)–(B.5) (with $U, V, [D,\cdot]$ instead of u, v, d). From these and the Leibniz rule (also taking into account unitarity of the generators u, v), it is obvious that $[D, U]$ and $[D, V]$ generate $\pi(\Omega^1(\mathcal{O}_\theta))$ as a left (or right) $\mathcal{O}_\theta$-module. To prove that it is a freely generated left module, assume $P[D, U] + Q[D, V] = 0$ with $P, Q \in \pi \oplus \pi(\mathcal{O}_\theta)$. It follows from (B.7) and (B.8) that

$$[D, U](e^{\pm}_{kl}) = \pm \mathrm{i}\sqrt{2\pi}\, e^{\mp}_{k+1,l}, \tag{B.9}$$

$$[D, V](e^{\pm}_{kl}) = \mathrm{e}^{2\pi \mathrm{i} k\theta}\sqrt{2\pi}\, e^{\mp}_{k,l+1}. \tag{B.10}$$

Therefore, terms in $P[D, U]+Q[D, V]$ can only compensate if they contain the same overall number of U and V. This means that it is sufficient to consider terms of the form $\alpha = pU^n V^{m+1}[D, U] + qU^{n+1}V^m[D, V]$, $p, q \in \mathbb{C}$. Acting on e^+_{kl}, we obtain

$$\alpha e^+_{kl} = \sqrt{2\pi}(\mathrm{e}^{2\pi \mathrm{i}(k+1)(m+1)\theta}\mathrm{i}p + \mathrm{e}^{2\pi \mathrm{i} km\theta}q)e^-_{k+n+1,l+m+1} = 0,$$

which is equivalent to

$$p\,\mathrm{e}^{2\pi \mathrm{i}(k+m+1)} + q = 0.$$

Since this should be true for all k, it follows that $p = q = 0$.

(ii) Differentiating the relations (B.1) and (B.2) gives immediately $\mathrm{d}u\,\mathrm{d}u = \mathrm{d}v\,\mathrm{d}v = \mathrm{d}u^*\,\mathrm{d}u^* = \mathrm{d}v^*\,\mathrm{d}v^* = 0$. Analogously, (B.3) leads to (B.6). Thus, we know already that $\mathrm{d}u\,\mathrm{d}v$ generates the two forms as a left (or right) $\mathcal{A}_\theta$-module. It remains to show that it is freely generated.

Let us recall that

$$\Omega^2_D(\mathcal{O}_\theta) \simeq \frac{\pi(\Omega^2(\mathcal{O}_\theta))}{\pi(\mathrm{d}J^1)},$$

where $J^1 = \ker\pi \cap \Omega^1(\mathcal{O}_\theta)$. Thus, any relation true in $\pi(\Omega^2(\mathcal{O}_\theta))$ is also true in $\pi(\Omega^2_D(\mathcal{O}_\theta))$. Now, a similar argument as in the proof of Proposition 4 (see [26]) shows that: (i) implies that J^1 coincides with the $\mathcal{O}_\theta$-subbimodule generated the elements corresponding to the relations (B.1)–(B.5). It follows that $\mathrm{d}J^1$ is a finite sum of elements of the form $a\,db\,c$ with $a, c \in \mathcal{O}_\theta$ and b one of the elements (B.1)–(B.5). Now, one shows by a direct computation that $\pi(db) \in \pi(\mathcal{O}_\theta)$ if b is one of the elements (B.1)–(B.5), whereas $\pi(db) = 0$ if b is one of the remaining elements. It follows that $\pi(\mathrm{d}J^1) = \pi(\mathcal{O}_\theta)$. It remains to show that from $\pi(a)[D, U][D, V] \in \pi(\mathcal{O}_\theta)$ it follows that $a = 0$. From the above formulae, it is now immediate that any element of the algebra acts on the $e^{\pm}_{kl}$ in a way not depending on $+$ or $-$, $\pi(a)e^{\pm}_{kl} = \sum \lambda_{ij}e^{\pm}_{ij}$, λ_{ij} independent on $+$ or $-$. On the other hand,

$$[D, U][D, V]e^{\pm}_{kl} = \pm 2\pi\mathrm{i}\,\mathrm{e}^{2\pi \mathrm{i} k\theta}e^{\pm}_{k+1,l+1},$$

from which (ii) follows immediately. □

Remark B.1. As in Remark 4, we can construct a topological version $\Omega_D(\mathcal{A}_\theta)$ of this calculus (using seminorms $q_n(\sum a_{kl}u^k v^l) = \sup_{kl}(1 + |k|^n + |l|^n)|a_{kl}|$). A comparison with the results of Strohmaier [30] shows that this gives indeed the calculus $\Omega_D(\mathcal{A}_\theta)$ of the spectral triple $(\mathcal{A}_\theta, \mathcal{H}, D)$.

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